MA26200: EXAM I **October 4, 2016**

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NAME: ____________________ **PUID:**____________________

Section Number: ____________________ **Class Time:**____________________

(1) No calculators are allowed.

(2) No portable electronic devices.

(3) There are 11 problems. Each problem is worth 11 points.

(4) The score is accumulative and the maximum is 110.

1 C

2 E

3 B

4 C

5 A

6 E

7 A

8 B

9 A

10 D

11 B

1. If $y' = \dfrac{3(x+1)^2}{y}$ and $y(-1) = 2$, then $y(0) =$

A. 3

B. $\sqrt{3}$

C. $\sqrt{6}$

D. 8

E. $\sqrt{10}$

Separable

$$\int y\,dy = \int 3(x+1)^2\,dx + C$$

$$\frac{1}{2}y^2 = (x+1)^3 + C$$

$$\frac{1}{2}2^2 = \frac{1}{2}y^2(-1) = (-1+1)^3 + C \Rightarrow C = 2$$

$$y^2 = 2(x+1)^3 + 4$$

$$y(0) = \sqrt{2+4} = \sqrt{6}$$

Q: Why don't we take $y = -\sqrt{2(x+1)^3+4}$?

A: $y' = \dfrac{3(x+1)^2}{y} \Rightarrow y \neq 0$

Since y is continuous, y take either positive value or negative value only

Because $y(-1) = 2 > 0$ y takes only positive value.

2. The solution of $\frac{dy}{dx} = \frac{x^2}{3y^2} + \frac{y}{x}$ satisfying $y(1) = 2$ is

A. $y^3 = \ln x + 2$

B. $y = x(\ln x)^{\frac{1}{3}} + 2$

C. $y^3 = x^3 \ln x + 2$

D. $y^3 = \ln x + 8$

E. $y^3 = x^3 \ln x + 8x^3$

$$\frac{dy}{dx} - \frac{y}{x} = \frac{x^2}{3y^2} \qquad \text{Bernoulli} \quad n = -2.$$

$$3y^2 \frac{dy}{dx} - \frac{1}{x} 3y^3 = x^2 \qquad \text{Let } u = y^3$$

$$\frac{du}{dx} - \frac{3}{x} u = x^2 \qquad I(x) = e^{\int -\frac{3}{x} dx} = \frac{1}{x^3}$$

$$\left(\frac{1}{x^3} u \right)' = \frac{1}{x^3} x^2 = \frac{1}{x}$$

$$\frac{1}{x^3} u = \ln|x| + C$$

$$\frac{y^3}{x^3} = \ln x + C \Rightarrow y^3 = x^3 \ln x + Cx^3$$

$$8 = y^3(1) = 1^3 \ln 1 + C \Rightarrow C = 8$$

$$y^3 = x^3 \ln x + 8x^3$$

Remark: We can remove the absolute value in $\ln|x|$, because

$\frac{y}{x} \Rightarrow x \neq 0 \Rightarrow x$ is either positive or negative.

Then $y(1) = 2 \Rightarrow x$ takes only positive value.

3. The general solution to the differential equation

$$(y\cos x + 2xe^y)dx + (\sin x + x^2e^y - \sec y\tan y)dy = 0$$

is

A. $-y\cos x + \dfrac{3x^2}{3}e^y - \cos x = C$

B. $y\sin x + x^2e^y - \sec y = C$

C. $y\sin x + x^2e^y - \tan y = C$

D. $\dfrac{y^2}{2}\cos x + 2xe^y - \sec x = C$

E. $\dfrac{y^2}{2}\cos x + 2xe^y - \tan x = C$

$M = y\cos x + 2xe^y \qquad N = \sin x + x^2e^y - \sec y\tan y$

$\partial_y M = \cos x + 2xe^y = \partial_x N \qquad$ Exact

$\phi = \int M\,dx + h(y) = y\sin x + x^2e^y + h(y)$

$\partial_y \phi = \sin x + x^2e^y + h'(y) = N = \sin x + x^2e^y - \sec y\tan y$

$\Downarrow$

$h'(y) = -\sec y\tan y$

$\Downarrow$

$h(y) = -\sec y$

$y\sin x + x^2e^y - \sec y = C$

4. A $100\,\ell$ tank initially contains $10\,kg$ of salt dissolved in $50\,\ell$ of water. Brine containing $1\,kg/\ell$ of salt flows into the tank at the rate $2\,\ell/\,\text{min}$, and the well-stirred mixture flows out of the tank at the rate $1\,\ell/\,\text{min}$. Which of the following describes $A(t)$, the amount of salt in the tank at time t before the tank becomes full?

A. $\frac{d}{dt}A + \frac{A}{10+t} = 2, \quad A(0) = 0.$

B. $\frac{d}{dt}A + \frac{A}{50+t} = 1, \quad A(0) = 0.$

C. $\frac{d}{dt}A + \frac{A}{50+t} = 2, \quad A(0) = 10.$

D. $\frac{d}{dt}A + \frac{A}{20+t} = 1, \quad A(0) = 10.$

E. $\frac{d}{dt}A + \frac{A}{100+t} = 2, \quad A(0) = 10.$

$A(0) = 10 \qquad V(0) = 50$

$r_1 = 2 \qquad C_1 = 1$

$r_2 = 1$

$V = (2-1)t + 50 = t + 50$

$$\frac{dA}{dt} = r_1 C_1 - r_2 C_2 = 2 - 1 \cdot \frac{A}{V} = 2 - \frac{A}{t+50}$$

$$\begin{cases} \dfrac{dA}{dt} + \dfrac{A}{50+t} = 2 \\ A(0) = 10 \end{cases}$$

5. If $A = \begin{bmatrix} 0 & 2 & -2 \\ 2 & 3 & 0 \\ 1 & 5 & -3 \end{bmatrix}$. What is the **sum** of the entries in the third row of A^{-1}?

A. $-\frac{5}{2}$

B. $\frac{5}{2}$

C. $\frac{3}{2}$

D. 1

E. 5

$$|A| = \begin{vmatrix} 0 & 2 & 0 \\ 2 & 3 & 3 \\ 1 & 5 & 2 \end{vmatrix} = -2 \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = -2$$

$$A_{13} = \begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} = 7$$

$$A_{23} = -\begin{vmatrix} 0 & 2 \\ 1 & 5 \end{vmatrix} = 2$$

$$A_{33} = \begin{vmatrix} 0 & 2 \\ 2 & 3 \end{vmatrix} = -4$$

$$\frac{A_{13} + A_{23} + A_{33}}{|A|} = \frac{7 + 2 - 4}{-2} = -\frac{5}{2}$$

6. Find all the values of k for which the system

$$\begin{cases} kx + y + z & = 1 \\ 3x + (k+2)y - z & = 5 \\ 2x + 2y + 2z & = k+1 \end{cases}$$

has no solution.

A. $k = 1$

B. $k = 1, -3$

C. $k \neq 1, -3$

D. $k \neq 1$

E. $k = -3$

$$D = \begin{vmatrix} k & 1 & 1 \\ 3 & k+2 & -1 \\ 2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} k-1 & 0 & 1 \\ 4 & k+3 & -1 \\ 0 & 0 & 2 \end{vmatrix} = (k-1)(k+3)$$

$\Rightarrow k = 1$ or -3

When $k = 1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 3 & 3 & -1 & 5 \\ 2 & 2 & 2 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \infty \text{ many solutions}$$

When $k = -3$

$$\left[\begin{array}{ccc|c} -3 & 1 & 1 & 1 \\ 3 & -1 & -1 & 5 \\ 2 & 2 & 2 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 3 & -1 & -1 & 5 \\ 0 & 0 & 0 & 6 \end{array}\right] \quad \text{No solution.}$$

8. Determine which one of the following expressions is the general solution to the inhomogeneous system of equations

$$\begin{cases} x_1 + x_2 - 2x_3 + 4x_4 &= 5 \\ 2x_1 + 2x_2 - 3x_3 + x_4 &= 3 \\ 3x_1 + 3x_2 - 4x_3 - 2x_4 &= 1 \end{cases}$$

A. $\begin{bmatrix} -9 \\ 0 \\ -7 \\ 0 \end{bmatrix}$

B. $\begin{bmatrix} -9 \\ 0 \\ -7 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 10 \\ 0 \\ 7 \\ 1 \end{bmatrix}$

C. $\begin{bmatrix} -11 \\ 2 \\ -7 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

D. $s \begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \\ -4 \end{bmatrix}$

E. No solution.

$$\left[\begin{array}{cccc|c} 1 & 1 & -2 & 4 & 5 \\ 2 & 2 & -3 & 1 & 3 \\ 3 & 3 & -4 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & -10 & -9 \\ 0 & 0 & 1 & -7 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

free variables

$$X = \begin{bmatrix} -9 \\ 0 \\ -7 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 10 \\ 0 \\ 7 \\ 1 \end{bmatrix}$$

9. Let C_{ij} be the cofactor of the element a_{ij} of the matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ with $\det(A) = 5$. Then the value of the expression $a_{11}C_{11} + a_{12}C_{12} - a_{21}C_{21} - a_{22}C_{22}$ is equal to

A. 0

B. 5

C. 10

D. 15

E. Undetermined by the information given above.

$a_{11} C_{11} + a_{12} C_{12} = |A| = 5$

$a_{21} C_{21} + a_{22} C_{22} = |A| = 5$

10. A is a 3×3 matrix and $AX = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ has no solutions. Then A satisfies

A. A is nonsingular.

B. $\det(A) \neq 0$.

C. The homogeneous system $AX = 0$ only has trivial solution.

D. The homogeneous system $AX = 0$ has infinitely many solutions.

E. None of the above.

$AX = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow A$ singular $\Leftrightarrow |A| = 0$

$\Leftrightarrow AX = 0$ has non-trivial solutions.

11. For an $n \times n$ matrix A, which of the following are true?

(i) $A + A^T$ is a symmetric matrix, and $A - A^T$ is a skew-symmetric matrix.

(ii) If A is both symmetric and skew-symmetric, then it is the zero matrix.

(iii) If n is even and A is skew-symmetric, then A is singular.

A. only (i)

B. only (i) and (ii)

C. only (ii) and (iii)

D. only (i), (ii) and (iii)

E. None of the above.

(i) $(A + A^T)^T = A^T + (A^T)^T = A^T + A$ sym

$(A - A^T)^T = A^T - (A^T)^T = A^T - A$

$= -(A - A^T)$ skew - sym

(ii) $A^T \overset{\text{sym}}{=} A$

skew sym $\rightarrow$ $\| \; -A$

$\Rightarrow A = -A \Rightarrow A = 0$

(iii) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is skew - sym.

But $|A| = 1$ invertible.